

Partial Wave Formalism and S -wave parametrization *

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Abstract

The formalism of Partial Wave Analysis is presented. Explicit formulae for amplitudes are given. The extended maximum likelihood method is described. $\pi\pi$ scalar-isoscalar parametrization of is discussed.

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1 Details of the Isobar model

For fixed center of mass energy $\sqrt{s} =$, the cross section of 3π production is a function of seven kinematical variables. In our analysis these are chosen as: the momentum transfer from the target to the recoiling nucleon, t , the invariant mass of the three pions, M and the invariant mass squared of a dipion sub-system, s_{ij} . Two out of the remaining four angular variables, $\Omega_{ij} = (\theta_{ij}, \phi_{ij})$, describe orientation of the dipion sub-system the Godfrey-Jackson frame, defined as the rest of the 3π system with the z -axis along the direction of the beam and the y axis perpendicular to the production plane formed by the recoiling nucleon and the total three-momentum of the three pions. The other two angles, $\Omega_i = (\theta_i, \phi_i)$ specify orientation of one of the two pions in the dipion-system and are defined in the helicity frame of the dipion sub-system, *i.e.* in the rest frame of the dipion with the $-z$ -axis along the bachelor, recoiling pion. The measured cross-section normalized to the observed number of events, can be written as,

$$d\sigma = \eta(t, M, s_{ij}, \Omega_{ij}, \Omega_k) I(t, M, s_{ij}, \Omega_{ij}, \Omega_k). \quad (1)$$

Here η describes detector acceptance, and the intensity I is determined from the spin-dependent production amplitudes,

$$I = \sum_{\lambda', \lambda} |A_{\lambda'\lambda}(t, M, s_{ij}, \Omega_{ij}, \Omega_k)|^2 \quad (2)$$

where, λ, λ' refer to the target and recoil helicity, respectively. In the absence of polarizations information we are forced to use spin coherence, or assume that a single (helicity non-flip) amplitude dominated production [?, ?], The aim of the partial wave analysis is use the data to extract the dependence of the production amplitude on a subset of kinematical variables *e.g.* M and t), under some assumed behavior on the other variables. In our analysis this dependence is determined by the isobar model in which the three-pion final state is reached via sequential two-body decays, with the interacting two-body intermediate states parametrized in terms of Breit-Wigner propagators. The exception is made for the S -wave two-body isobars. The s -wave parametrization and the limitations of the isobar model are discussed in the following section. For the isobar model the intensity is written as,

$$I(\tau) \propto \left| \sum_i a_i A_i \right|^2 \quad (3)$$

where a_i, A_i are production and decay amplitudes and the goal of PWA is to determine the quantities a_i . The production amplitudes are given by,

$$A_{l s_{12} J M P} = \sqrt{(2l+1)(2s_{12}+1)} \sum_{\lambda_1} D_{M\lambda_1}^{J*}(\phi_{12}, \theta_{12}, 0) D_{\lambda_{10}}^{s_{12}*}(\phi_1, \theta_1, 0) \langle l 0 s_{12} \lambda_1 | J \lambda_1 \rangle F_l(p_{12}) F_{s_{12}}(p_1) \Delta(s_{12}) \quad (4)$$

where l denotes the angular momentum quantum number between two-pion, labeled here as particles 1 and 2 isobar and the bachelor pion labeled as particle 3. $m_{\pi\pi} = \sqrt{s_{12}}$ is the invariant mass of isobar, p_{12} is the isobar momentum in the GJ frame and p_1 is the breakup momentum of the pions

in the isobar rest frame. Finally $\Delta(m_{\pi\pi})$ is Breit-Wigner propagator for the isobar with nominal mass w_0 and spin s_1 ,

$$\Delta(w) = \frac{w_0 \Gamma_0}{w_0^2 - w^2 - iw_0 \Gamma(w)} \quad (5)$$

and

$$\Gamma(w) = \Gamma_0 \frac{w_0}{w} \frac{q}{q_0} \frac{F_{s_1}^2(q)}{F_{s_1}^2(q_0)}. \quad (6)$$

q and q_0 are breakup momenta. In our case $q = p_3$ and

$$q_0 = \frac{\sqrt{\lambda(w_0^2, m_3^2, m_4^2)}}{2w_0} \quad (7)$$

where

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (8)$$

$F_s(q)$'s denote angular momentum barrier factors derived by von Hippel and Quigg. Explicitly, the first few barrier factors are

$$F_0(q) = 1 \quad (9)$$

$$F_1(q) = \sqrt{\frac{2z}{z+1}} \quad (10)$$

$$F_2(q) = \sqrt{\frac{13z^2}{(z-3)^2 + 9z}} \quad (11)$$

$$F_3(q) = \sqrt{\frac{277z^3}{z(z-15)^2 + 9(2z-5)^2}} \quad (12)$$

$$F_4(q) = \sqrt{\frac{12746z^4}{(z^2 - 45z + 105)^2 + 25z(2z - 21)^2}} \quad (13)$$

where $z = (q/q_R)^2$ and $q_R = 0.1973(GeV/c)$ corresponding to $1fm$.

Since there are two identical particles in the final state, Bose symmetry requires

$$A_{ls_1 J M P} \rightarrow \sum_{perm} A_{ls_{ij} J M P} \quad (14)$$

where permutation refers to all identical particles.

There are additional symmetries of the production process which lead to the concept of reflectivity. Decay amplitudes to be used in the expression for intensity will be characterized by an additional quantum number, ε , the reflectivity, denoting properties of the amplitude under the reflection operator. [†] ε coincides with the naturality of the exchanged Regge trajectory and takes two values, +1 and -1. Consequently we define

$$A_{ls_1 J M P \varepsilon} = \theta(M) [A_{ls_1 J M P} - \varepsilon P^{J-M} A_{ls_1 J - M P}] \quad (15)$$

[†]by reflection we mean reflection w.r.t xz -plane of the GJ frame.

where

$$\theta(M) = \begin{cases} 1/\sqrt{2} & \text{for } M > 0 \\ 1/2 & \text{for } M = 0 \\ 0 & \text{for } M < 0 \end{cases} \quad (16)$$

For the the 3π case the explicit form of the amplitudes is given by,

$$A_{ls_{12}JMP\varepsilon} = \left[\sqrt{(2l+1)(2s_{12}+1)} \sum_{\lambda} B_{m\lambda}^{J\varepsilon\star}(\phi_{12}, \theta_{12}, \phi_1) d_{\lambda 0}^{s_{12}}(\theta_1) \langle l 0 s_{12} \lambda | J \lambda \rangle \right] F_l(p_{12}) F_{s_{12}}(p_1) \Delta(s_{12}) \quad (17)$$

where $d_{\lambda 0}^{s_1}$ denotes Wigner d -functions and $B_{m\lambda}^{J\varepsilon\star}(\phi_{12}, \theta_{12}, \phi_1)$ is given by

$$B_{m\lambda}^{J\varepsilon\star}(\phi_{12}, \theta_{12}, \phi_1) = \theta(m) [D_{m\lambda}^{J\star}(\phi_{12}, \theta_{12}, \phi_1) + \varepsilon D_{m\lambda}^J(\phi_{12}, \theta_{12}, \phi_1)] \quad (18)$$

At this point we are ready write an expression for the intensity in terms of production and decay amplitudes

$$I(m_{3\pi}, t, \tau) = \sum_{\varepsilon} \left| \sum_b a_b^{\varepsilon}(m_{3\pi}, t) A_b^{\varepsilon}(\tau) \right|^2 \quad (19)$$

The production amplitude is written as explicitly dependent on $m_{3\pi}$, the invariant mass of the 3π system and t , the four momentum transfer. τ is set of kinematic variables describing decay of the resonance. In our case of 3π final state

$$\tau = \{\phi_{12}, \theta_{12}, \phi_1, \theta_1, s_{12}\}. \quad (20)$$

The indices $b = \{l, s_{12}, J, M, P, v\}$ and ε enumerate partial waves.

Finally, the goal of PWA is to find the production amplitudes $a_b^{\varepsilon}(m_{3\pi}, t)$ as a function of invariant mass $m_{3\pi}$ and four-momentum transfer t . The techniques used to determine these complex numbers is discussed below.

1.1 Extracting production amplitudes - likelihood method

In this section we list the formulas necessary to implement a PWA fitting program. The experimental data sample is binned by invariant mass $m_{3\pi}$ and t . The likelihood function for an $(m_{3\pi}, t)$ bin is defined as

$$\mathcal{L} \propto \left[\frac{\bar{n}^n}{n!} e^{-\bar{n}} \right] \prod_{i=0}^n \left[\frac{I(\tau_i)}{\int d\tau p q \eta(\tau) I(\tau)} \right] \quad (21)$$

where the expression in the first bracket is the probability of observing n events when the expected number is \bar{n} ,

$$\bar{n} = \int d\tau p q \eta(\tau) I(\tau) \quad (22)$$

where $\eta(\tau)$ is the experimental acceptance.

It is more computationally convenient to compute the logarithm of the likelihood

$$\ln \mathcal{L} \propto \sum_{i=0}^n \ln I(\tau_i) - \int d\tau p q \eta(\tau) I(\tau). \quad (23)$$

This expression is to be maximized to determine the relevant amplitudes. For this reason terms not dependent on I have been neglected in equation ???. Inserting the expression for $I(\tau)$ from equation ??? into equation ??? yields

$$\ln \mathcal{L} \propto \sum_{i=0}^n \ln I(\tau_i) - \sum_{\epsilon} \sum_{b,b'} a_b^{\epsilon} a_{b'}^{*\epsilon} \int d\tau p q \eta(\tau) A_b^{\epsilon}(\tau) A_{b'}^{*\epsilon}(\tau). \quad (24)$$

The squared summation of equation ??? has been expanded. The remaining integral is approximated with a sum over accepted Monte Carlo events

$$\int d\tau p q \eta(\tau) A_b^{\epsilon}(\tau) A_{b'}^{*\epsilon}(\tau) \approx \frac{1}{N_{raw}} \sum_i^{N_{acc}} A_b^{\epsilon}(\tau_i) A_{b'}^{*\epsilon}(\tau_i). \quad (25)$$

For computational convenience we introduce the following quantities

$$\eta_{acc} \equiv \frac{N_{acc}}{N_{raw}} \quad (26)$$

$$\Psi_{acc}^{\epsilon}(b; b') \equiv \frac{1}{N_{acc}} \sum_i^{N_{acc}} A_b^{\epsilon}(\tau_i) A_{b'}^{*\epsilon}(\tau_i) \quad (27)$$

so that

$$\frac{1}{N_{raw}} \sum_i^{N_{acc}} A_b^{\epsilon}(\tau_i) A_{b'}^{*\epsilon}(\tau_i) = \eta_{acc} \Psi_{acc}^{\epsilon}(b; b') \quad (28)$$

where N_{acc} (N_{raw}) is the number of accepted (raw) MC events in a given $m_{3\pi}$ and t bin.

The expression for log-likelihood can be rewritten in the following way

$$\ln \mathcal{L} \propto \sum_{i=0}^{N_{data}} \ln I(\tau_i) - \eta_{acc} \sum_{\epsilon} \sum_{b,b'} a_b^{\epsilon} a_{b'}^{*\epsilon} \Psi_{acc}^{\epsilon}(b; b'). \quad (29)$$

Additionally one may redefine production amplitudes, so that they are of the same order for all mass bins, as follows

$$a_{lm}^{\epsilon} = \sqrt{\frac{N_{data}}{\eta_{acc}}} \tilde{a}_b^{\epsilon}. \quad (30)$$

Such scaling assures that amplitudes \tilde{a}_b^{ϵ} in bins with small number of observed events are, approximately, of the same order as amplitudes in bins with large number of observed events.

Finally

$$\ln \mathcal{L} \propto \sum_{i=0}^{N_{data}} \ln I(\tau_i) - N_{data} \sum_{\epsilon} \sum_{b,b'} \tilde{a}_b^{\epsilon} \tilde{a}_{b'}^{*\epsilon} \Psi_{acc}^{\epsilon}(b; b'). \quad (31)$$

Usually one wants to add background to the parameterization of the intensity to account for backgrounds in the data. Incomplete or imperfect understanding of the PWA model can also, at

least partially, be subsumed into this background. Background can be considered is a measure of both the quality of the data sample and the quality of the theoretical model that we use.

Adding a single parameter for the background, B the intensity $I(\tau)$ can be written as

$$I^B(\tau) = \sum_{\epsilon} \sum_{b,b'} a_b^{\epsilon} a_{b'}^{\star\epsilon} A_b^{\epsilon}(\tau) A_{b'}^{\star\epsilon}(\tau) + B^2 \quad (32)$$

and equation (??) becomes

$$\ln \mathcal{L} \propto \sum_{i=0}^{N_{data}} \ln I^B(\tau_i) - N_{data} \sum_{\epsilon} \sum_{b,b'} \tilde{a}_b^{\epsilon} \tilde{a}_{b'}^{\star\epsilon} \Psi_{acc}^{\epsilon}(b; b') + \tilde{B}^2 \quad (33)$$

where

$$B = \sqrt{\frac{N_{data}}{\eta_{acc}}} \tilde{B}. \quad (34)$$

The parameters a_b^{ϵ} and B can now be determined using a standard minimization package, for example MINUIT.

Once the production amplitudes a_b^{ϵ} have been determined from the fit the number of acceptance corrected events in a particular partial wave can be calculated as

$$N_{ev}(b, \epsilon) = |\tilde{a}_b^{\epsilon}|^2 \frac{N_{data}}{\eta_{acc}} \Psi_{raw}^{\epsilon}(b, b) \quad (35)$$

Quite often we want to know number of events not just in a single wave but in several waves. In such a case number of events is given by

$$N_{ev}(L_{b,\epsilon}) = \sum_{\epsilon} \sum_{L_{b,\epsilon}, L'_{b,\epsilon}} \tilde{a}_{L_{b,\epsilon}}^{\epsilon} \tilde{a}_{L'_{b,\epsilon}}^{\star\epsilon} \Psi_{raw}^{\epsilon}(L_{b,\epsilon}; L'_{b,\epsilon}) \quad (36)$$

where $L_{i,\epsilon}$ enumerates waves for which we wish to find the total number of events.

1.2 Word of caution on normalization integrals

To approximate integrals of the type $\int d\tau pq A_b^{\epsilon}(\tau) A_{b'}^{\star\epsilon}(\tau)$ in the following way

$$\int d\tau pq A_b^{\epsilon}(\tau) A_{b'}^{\star\epsilon}(\tau) \approx \frac{1}{N_{raw}} \sum_i^{N_{raw}} A_b^{\epsilon}(\tau_i) A_{b'}^{\star\epsilon}(\tau_i). \quad (37)$$

we have to make sure that RawMC data sample was generated with distribution $d\tau pq$. If this is not the case, the above approximation is not valid. Similarly for the following approximation to be valid

$$\int d\tau pq \eta(\tau) A_b^{\epsilon}(\tau) A_{b'}^{\star\epsilon}(\tau) \approx \frac{1}{N_{raw}} \sum_i^{N_{acc}} A_b^{\epsilon}(\tau_i) A_{b'}^{\star\epsilon}(\tau_i). \quad (38)$$

one has to use properly distributed RawMC data sample (i.e. events should be distributed according to $d\tau pq$) and from it generate AcceptedMC sample with events distributed according to $d\tau pq \eta(\tau)$.

1.3 Isobar parameters used in the PWA fit

In the PWA formalism that we have just described, one is forced to use fixed parameters of the isobars. It is therefore beneficial to list all the isobar parameters that are used in the PWA fits. Parameterization of the S -wave ($I = 0$) is described in detail in the separate note [reference]

For completeness, the table below in addition to isobars contains also masses of the pions. Note that not isobars are used in the 3π fit.

Table 1: Isobar parameters

Isobar name	J^{PC}	Mass [GeV]	Width [GeV]
ρ_3	3^{--}	1.691	0.215
$f_2(1275)$	2^{++}	1.275	0.185
$f_0(980)$	0^{++}	0.974	0.055
$f_0(1300)$	0^{++}	1.300	0.400
$f_0(1520)$	0^{++}	1.520	0.120
ρ^0	1^{--}	0.7699	0.1512
$\rho(1450)$	1^{--}	1.452	0.310
$\pi^+(\pi^-)$	0^-	0.1395679	
π^0	0^{-+}	0.1349764	

2 Details of the $\pi\pi$ S -wave amplitudes

As discussed above, in the isobar model the 2π partial wave amplitude enters via the isobar amplitude

$$\Delta(s = m_{\pi\pi}^2) \equiv D^{-1}(s), \quad (39)$$

This amplitude has right hand unitarity cuts, which are the same as in the corresponding 2π scattering amplitude, T , which in turn can be parametrized as,

$$T(s) = D^{-1}(s)N(s) \quad (40)$$

with the numerator function (driving term) containing potential or crossed channel, left hand cuts. The functions $D(s) = D_{ij}(s)$ and $N(s) = N_{ij}(s)$ are in general matrices in the space of channels coupled to the 2π system. Analyticity of the partial wave amplitude, T , leads to dispersion relations for the functions N and D . The N over D parameterization of two-body partial wave amplitudes enables to fulfill requirements of unitarity, however, if the two-particle state is a part of a larger final state in which other binary interaction are possible, the parameterization of Eq. (??) violates two-body unitarity in the final state [?]. In the 3π system there has been previous investigations of corrections to the isobar model but no significant change in the three-body partial wave amplitudes have been found [?, ?]. Nevertheless such parameterization should be considered, in particular in view of the large statistics data nowadays available.

In summary, in the isobar model the analytical structure of the two-particle subchannel partial waves is identical to that of the corresponding scattering amplitude. This structure is typically parametrized using *i*) dispersion relations without elementary particle poles, *ii*) dispersion relations with elementary particle poles – the so called CDD poles, or *iii*) the K -matrix approximation. In our analysis we used three different parameterizations, each representing one of these three cases.

2.1 Parameterization based on dispersion relations without CDD poles

We refer to this parameterization as the V parameterization. As an example we employ the model of the Valencia group [?] where two channels, $\pi\pi$, and $K\bar{K}$, referred to as 1 and 2, respectively are used. The elastic amplitude T is written as the N over D ratio,

$$T = D^{-1}(s)N(s) \quad (41)$$

with the denominator containing the right hand and N left hand cuts, respectively. The driving term for the scattering amplitude is given by the $O(p^2)$ chiral Lagrangian. Explicitly,

$$N_{11} = \frac{2s - m_\pi^2}{2f_\pi^2}, \quad N_{12} = N_{21} = \sqrt{3}\frac{s}{4f_\pi^2}, \quad N_{22} = 3\frac{s}{4f_\pi^2}, \quad (42)$$

where $f_\pi = 93$ MeV is the pion decay constant. Analyticity of the partial wave amplitude, T , leads to dispersion relations which relate the two functions N and D . Since for large s , $N = O(s)$ subtractions are needed in the dispersion relation for D [?]. This introduces a parameter which is fixed by fitting the $\pi\pi$ and $K\bar{K}$ S -wave data. For $N(s)$ diverging as $s \rightarrow \infty$ it is possible to make $N(s) \rightarrow \text{cons.}$ by dividing both N and D by an appropriate polynomial in s , *i.e.* without changing their analytical structure. This in general introduces poles in D –the CDD poles. Since the CDD poles in D are accompanied by nearby zeros they produce resonant-like behavior of the scattering amplitude and are interpreted as due to elementary, QCD particles (ultimately analyticity of the full amplitude, with all partial waves included, removes the apparent arbitrariness in the number of CDD poles). In the case of N given by Eq. (??) this leads to the σ -meson. The denominator function satisfying a subtracted dispersion relation is given by,

$$D(s) = I + N\text{Re}G - iN\rho \quad (43)$$

where $\rho(s) = \rho_{ij}(s) = \rho_i(s)\delta_{ij}$ and

$$\rho_i(s) = q_i/(8\pi\sqrt{s})\theta(s/4 - m_i^2), \quad q_i = \sqrt{s/4 - m_i^2} \quad (44)$$

is the two-body phase space, and $\text{Re}G \equiv g_i(s)\delta_{ij}$ is the real part of the subtracted Chew-Mandelstam function [?] with

$$g_i(s) = \frac{a_i}{(4\pi)^2} + \begin{cases} \frac{1}{(4\pi)^2} \sqrt{1 - \frac{4m_i^2}{s}} \log \frac{1 + \sqrt{1 - \frac{4m_i^2}{s}}}{1 - \sqrt{1 - \frac{4m_i^2}{s}}} \text{ for } s > 4m_i^2 \\ \frac{2}{(4\pi)^2} \sqrt{\frac{4m_i^2}{s} - 1} \arctan \frac{1}{\sqrt{\frac{4m_i^2}{s} - 1}} \text{ for } 0 < s < 4m_i^2 \end{cases} \quad (45)$$

The subtraction constants a_i are determined by fitting the combined $\pi\pi$ and $K\bar{K}$ data, yielding $a_1 = -5.22$, $a_2 = -2.60$. The denominator function, $\Delta(s) = D^{-1}$ can also be expressed in terms of the K matrix defined by

$$K^{-1} = T^{-1} + i\rho = N^{-1} + ReG \quad (46)$$

In the K -matrix approximation, which will be described later, the Chew-Mandelstam function is usually neglected, *i.e.* $K = N$. In terms of the K matrix the denominator function is given by

$$\Delta^{-1} = TN^{-1} = [1 - iK\rho]^{-1}KN^{-1} \quad (47)$$

In our PWA the matrix elements Δ_{11} and Δ_{12} were used. These correspond to production of $\pi\pi$ isobars via $\pi\pi$ or $K\bar{K}$ intermediate state, respectively.

2.2 Parameterization based on dispersion relations with CDD poles

To the two-body, $\pi\pi$ and $K\bar{K}$ channels we add a scalar-isoscalar and a scalar-vector CDD pole. The denominator function is written in terms of the 2×2 K matrix,

$$K = [1 + (N + N^P)ReG]^{-1}N \quad (48)$$

with the real part of the Chew-Mandelstam function given by Eq (??) with the two subtraction constants a_i obtained by re-fitting the combined $\pi\pi$ and $K\bar{K}$ data. As in the earlier parameterization the numerator function, N is chosen as the low-energy, two-meson scattering amplitude from the $O(p^2)$ chiral expansion. Here, however, we also add from factors to eliminate the large- s behavior, so that Eq. (??) is modified by,

$$N_{ij} \rightarrow g_i N_{ij}(Eq. (??)) g_j \quad (49)$$

with $g_i = \Lambda^2/(q_i^2 + \Lambda^2)$. The second driving term, N^P comes from the two added CDD poles. These are needed to account for the rise in the phase shifts at $\sqrt{s} = 1.3$ GeV and $\sqrt{s} = 1.5$ GeV. For N^P we use the following parameterization [?],

$$\begin{aligned} N_{11}^P &= \frac{3}{2}\alpha^2 \frac{s_0^0}{s_0^0 - s} + \frac{3}{2}\beta_1^2 \frac{s_0^8}{s_0^8 - s} \\ N_{12}^P &= N_{21}^P = \sqrt{3}\alpha^2 \frac{s_0^0}{s_0^0 - s} + \sqrt{3}\beta_1\beta_2 \frac{s_0^8}{s_0^8 - s} \\ N_{22}^P &= 2\alpha^2 \frac{s_0^0}{s_0^0 - s} + 2\beta_2^2 \frac{s_0^8}{s_0^8 - s} \end{aligned} \quad (50)$$

with

$$\begin{aligned} \alpha &= c \frac{s}{f_\pi^2} + d, \\ \beta_i &= e_i \frac{s}{f_\pi^2} + f_i \end{aligned} \quad (51)$$

The parameters a_i , Λ, c, d, e_i, f_i , and the positions of the CDD poles s_0^0 and s_0^8 were fitted to the combined $\pi - \pi$ and $K\bar{K}$ S -wave data and are given in Table ??.

a_1	-5.16
a_2	-2.54
Λ	0.88 GeV
$\sqrt{s_0^8}$	1.36 GeV
$\sqrt{s_0^0}$	1.55 GeV
c	0.02
d	-0.29
e_1	-0.21
f_1	1.48
e_2	0.01
f_2	-0.74

Table 2: Parameters of the S -wave amplitudes for the C -parameterization

As before, the two-pion propagator used in PWA can be expressed in terms of the K -matrix,

$$\Delta = D^{-1} = [1 - iK\rho]^{-1}KN^{-1} = TN^{-1} \quad (52)$$

For this parameterization, in our PWA we also used the matrix elements Δ_{11} and Δ_{12} .

2.3 Parameterization based on the K -matrix approximation

This parameterization uses the M -fit to the K -matrix of Au, Morgan and Pennington [?]. For completeness we list all parameters in Table ???. In this parameterization the Chew-Mandelstam function is ignored (*i.e.* $K = N$) and the K -matrix is chosen in the form,

$$K^{-1} = \frac{A}{s - s_0} + \frac{F}{s_1 - s} + \sum_{n=0}^4 C^n \left[\frac{s}{4m_K^2} - 1 \right]^n \quad (53)$$

The denominator function, as for other parameterizations is given by

$$\Delta = D^{-1} = [1 - iK\rho]^{-1}KN^{-1} = TN^{-1}, \quad (54)$$

however, the original E852 analysis used

$$\Delta = D^{-1} = [1 - iK\rho]^{-1}K = T \quad (55)$$

instead, and in our fits we have used this parameterizations to be consistent with the previous analysis. Physically, this assumes that the structure of the left hand cut of the production of pion (and kaon) pairs is identical to that of the two-body amplitudes. This is appropriate if production of the $\pi\pi$ sub-system proceeds via the pion exchange *e.g.* as in the case of $\pi N \rightarrow \pi\pi N$ studied in Ref. [?]. This is for example expected for the Deck mechanism, which is known to be substantial. In our fits (and previous E852 analysis) only the $\pi\pi \rightarrow \pi\pi$, Δ_{11} amplitude was used.

Finally, in the original E852 parameterization, the $f_0(980)$ has been subtracted from Δ and added explicitly as a Breit-Wigner propagator. This is done by the following replacement,

s_0	-0.0074 GeV ²
s_1	0.983 GeV ²
A_{11}	2.25
$A_{12} = A_{21}$	0.30
A_{22}	-6.40
F_{11}	0.77
$F_{12} = F_{21}$	-0.06
F_{22}	0.0047
C_{11}^0	0.67
$C_{12}^0 = C_{22}^0$	-5.62
C_{22}^0	5.99
C_{11}^1	-6.34
$C_{12}^1 = C_{22}^1$	1.83
C_{22}^1	-10.23
C_{11}^2	-1.87
$C_{12}^2 = C_{22}^2$	3.32
C_{22}^2	2.34
C_{11}^3	-11.79
$C_{12}^3 = C_{22}^3$	-4.14
C_{22}^3	10.35
C_{11}^4	3.89
$C_{12}^4 = C_{22}^4$	-2.76
C_{22}^4	-7.91

Table 3: Parameters of the S -wave amplitudes for the M -parameterization. The matrix elements of A , F and C are in units of 10^{-3} GeV². The parameters differ from those of Ref. [?] by a factor 16π due to a difference in normalization of the two-body phase space used here.

$$\Delta_{11} \rightarrow \Delta_{11} - c \frac{w' \Gamma' \left(\frac{\sqrt{s}}{q} \right)}{w'^2 - s - iw' \Gamma'} \quad (56)$$

with $c = (-0.3743, 0.3197)$, $w' = 0.9837$ GeV and $\Gamma' = 37.6$ MeV.

The $f_0(980)$ amplitude is then added as a K -matrix amplitude with,

$$\Delta_{f_0}(s) = [1 - iK_{f_0}\rho]^{-1} K_{f_0}, \quad K_{f_0} = \frac{f^2}{w^2 - s} \quad (57)$$

and $w = 0.974$ GeV, $f^2 = 2.81$ GeV².

In Fig. 1 (panels labeled 1 \rightarrow 5) we plot the real and imaginary parts of the elastic, $\pi\pi \rightarrow \pi\pi$ components of the T matrix, T_{11} for all parameterizations described above. In the last panel we plot the magnitude $|\rho T| = |\exp(2i\delta_{\pi\pi}) - 1|/2$ for all five amplitudes.

References

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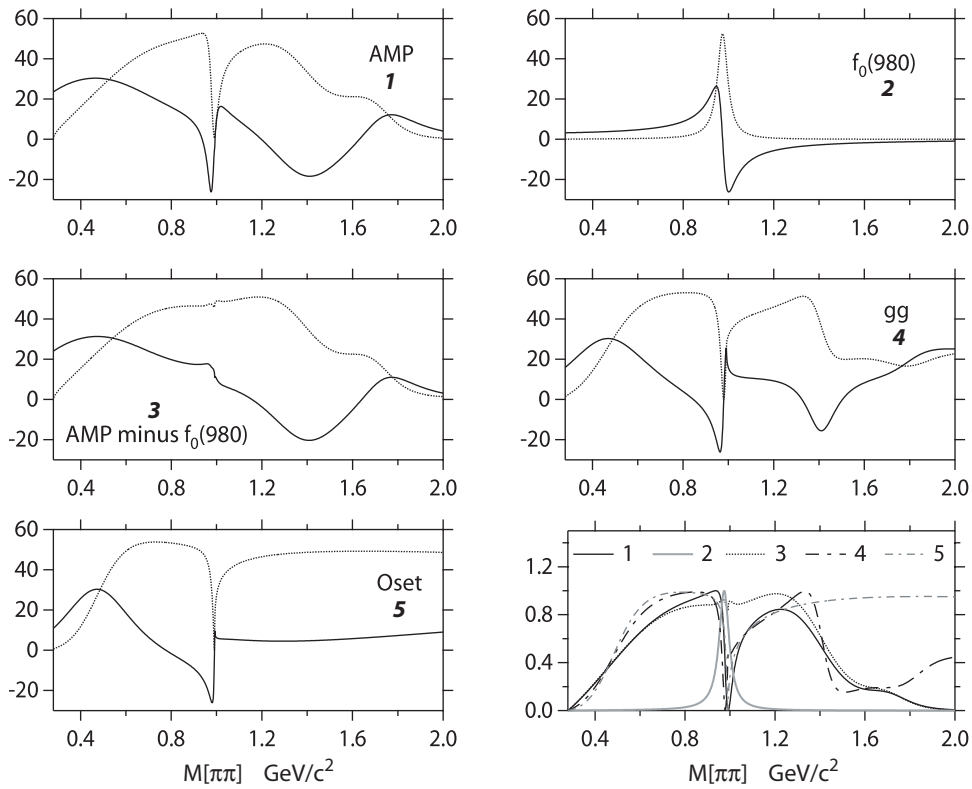


Figure 1: Real and imaginary parts of T_{11} for amplitudes described in text (panels 1 \rightarrow 5, (gg = with CDD poles, Oset = without CDD poles) and the magnitude of ρT for all amplitudes.

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